Pursuit–escape with distance-dependent delay

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Summary. We introduce a time delay into a simple pursuit–escape model. It is shown that when the delay is state-dependent, the complexity of the dynamics become exquisitely sensitive to the relative positions of the chaser and escapee.

Introduction

Issues related to chase and escape are vitally important for living organisms. Indeed for many whether they eat or are eaten depends critically on their ability to capture prey on the one hand, and their ability to escape predation on the other [1]. Human children hone their chase and escape skills through games and sports and then may have to depend on them as adults in life-threatening situations associated with crime and military actions. Mathematical interest was first attracted to chase–escape problems as early as the 5th century B.C. and over intervening centuries many interesting results have been obtained (for reviews see [2, 3, 4]).

Often overlooked in the analysis of chase–escape are the effects of time delays. Time delays arise, for example, because it takes the nervous system of the pursuer time to detect a change in speed and/or direction of the escapee and furthermore time to compute and implement a change in pursuit or escape strategy. Recent attention has focused on a human visuomotor tracking task known as virtual stick balancing [5, 6, 7, 8, 9]. In this task, the human operator's task is to keep the target (escapee) from escaping off the computer screen by moving the computer mouse (chaser) appropriately. The analogy to real stick balancing, say at the fingertip, is made by programming the movements of the target in a parametric potential centered at the target. In other words, when the position of the target and a dot controlled by the computer mouse are at the same location, this position is unstable and hence even the smallest perturbation causes the target to escape. Depending on the skill level of the human operator, a complex variety of dynamical behavior has been observed, including an oscillatory relation between the movements of the dot and target [9] as well as a situation in which the mouse makes intermittent corrective movements that exhibit power–law behavior [6, 7, 8].

A first step towards understanding the complex dynamical behavior exhibited by delayed pursuit–escape is to examine the effects of delay on a simpler pursuit–escape task whose dynamics have been well characterized by mathematical analyses. Here we introduce a state–dependent time delay into Hathaway’s classical dog–duck pursuit–escape task (see [4] for the case with no delay). In this problem the duck swims on a circular path and the problem is to determine the dog’s best strategy for pursuing and perhaps catching the duck. Our preliminary results demonstrate that the addition of a state–dependent delay can make the dynamics of this pursuit–escape task very complex. Moreover this complexity appears to be related to an exquisite sensitivity of the movements of the dog and the duck to their relative positions.

Figure 1: Positions of the chaser and the target.
Background

A duck swims on a circular path and is pursued by a dog who can swim $n$ times faster. Suppose that the initial position of the duck at time $t = 0$ is on the $x$-axis, $(a, 0)$. At $t > 0$ later the duck has swum through an arc of angle $\theta$, i.e. a distance $a\theta$, and hence the dog swims a distance of $s = na\theta$ to reach the location $(x, y)$. It is assumed that the dog executes a pure pursuit. In other words the tangent at $(x, y)$ to the dog’s pursuit curve passes through the duck’s instantaneous position. Define the angle made by the tangent line and the $x$-axis as $\varphi$ and the distance between dog and duck as $\rho$. Then it can be shown that the equation of the tangent line to the dog’s pursuit curve is

$$x \sin \varphi - y \cos \varphi = a \sin(\varphi - \theta)$$

and that the equation of the line normal to the tangent line that passes through $(x, y)$ is

$$x \cos \varphi + y \sin \varphi = a \cos(\varphi - \theta) - \rho$$

By differentiating these equations with respect to $\theta$ we obtained, after many steps, the system of differential equations

$$\rho \frac{d\varphi}{d\theta} = a \cos(\varphi - \theta)$$

$$\frac{d\rho}{d\theta} = a \sin(\varphi - \theta) - a \rho$$

These equations cannot be solved analytically. However, we can still make some understanding for the nature of the dog’s trajectory. For example, we can derive differential equation that describes the distance between dog and duck by defining $\phi = \varphi - \theta$ and

$$\rho \frac{d^2\rho}{dt^2} + a \rho \cos(\phi) = a^2 \cos^2 \phi$$

(1)

This equation means that the limiting behavior of $\rho$ as $t \to \infty$ the chord between the dog and duck rotates but its length remains unchanged, so that when

$$\frac{d\rho}{dt} = \frac{d^2\rho}{dt^2} = 0$$

we have

$$\rho = a \cos \theta$$

Further, we can show that the when $n \leq 1$, the dog cannot catch up the duck (when separated initially, of course), and that the trajectory of the dog will eventually become a circle with radius $R = na$.

Model

We introduce a state-dependent time delay into the Hathaway’s task. We assume that the tangent line of the dog’s pursuit curve points not to the duck’s present position as in the original model, but to its past position by a time delay $\tau$. If $\tau$ is constant, then the problem can be mapped to a difference of the duck’s initial position in the original problem. Thus the qualitative properties of the pursuit task are unchanged: a constant delay is equivalent to the introduction of a constant, then the problem can be mapped to a difference of the duck’s initial position in the original problem. Thus the equation of the tangent line to the dog’s pursuit curve is

$$x \sin \varphi - y \cos \varphi = a \sin(\varphi - \theta)$$

and that the equation of the line normal to the tangent line that passes through $(x, y)$ is

$$x \cos \varphi + y \sin \varphi = a \cos(\varphi - \theta) - \rho$$

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Results

Figure 2 compares the behavior of the dog–duck pursuit task as a function of $\tau$. Increases of $\tau_0$ lead to a variety of trajectories for the chaser with the escapee moving in the unit circular path including

- When $n < 1$, the dog cannot catch the duck, irrespective of the presence of delay (Fig. 2b without delay and Fig. 2e with delay).
- When $n > 1$ with no delay, the dog can catch the duck (Fig. 2a). However, with the delay increased beyond a critical value, the dog cannot catch the duck (Figs. 2c, 2d, 2f–h).
- With delay, trajectories of the dog can be quite complex (Figs. 2c–h).

The complexity of the movements of the dog shown in Figure 2 arise because $\tau$ is an increasing function of $\rho$. To illustrate consider the effect of a change in $\rho$, $\Delta\rho$, on the movements of the dog over a small time interval $dt$. If $\Delta\rho > 0$, then $\tau$ for the next step increases. This, in turn, decreases the dog’s precision for detecting the duck at the next $dt$. Note that since the duck is confined to move along a circle at constant velocity, the increase in $\tau$ does not necessarily imply that the dog gets further from the duck. On the other hand, if $\Delta\rho < 0$ then the velocity of the dog would point to the present
Figure 2: Examples of a circular chase (red) and escape (blue) with the distance dependent delays. The scale factor and speed ratio $[\tau_0, n]$ are given as (a) $[0, 1.01]$, (b) $[0, 0.5]$, (c) $[500, 1.01]$, (d) $[500, 1.5]$, (e) $[500, 0.5]$, (f) $[1050, 1.01]$, (g) $[700, 1.5]$, (h) $[1500, 1.01]$ with the duck’s velocity $v = 0.05$. (The period to round the unit circle is $T \approx 126$.)

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The movement. When the dog moves in the positive direction of velocity vector of the dog is written as the present position of the duck (\(a = 1\)), while the dog is located at \((x_c, y_c) = (r \cos \lambda, r \sin \lambda)\) (Figure 1). The distance \(\rho\) between the dog and the duck is

\[
\rho = \sqrt{(r \cos \lambda - 1)^2 + (r \sin \lambda)^2} = \sqrt{1 + r^2 - 2r \cos \lambda}.
\]

Here the time delay is given as \(\tau = \tau_0 \rho\). If we take the angular velocity of the duck as \(\omega\), its position at \(\tau\) past is given by \((x_T^f, y_T^f) = (\cos 2\pi \omega \tau, -\sin 2\pi \omega \tau)\). The dog at \((x_c, y_c)\) points its velocity vector to the position \((x_T^f, y_T^f)\). Therefore, the direction of velocity vector of the dog is written as

\[
(v_x, v_y) \equiv (\cos 2\pi \omega \tau - r \cos \lambda, -\sin 2\pi \omega \tau - r \sin \lambda).
\]

We can compute the position of the dog with this velocity vector \(dt\) later as

\[
(x_c', y_c') = (r \cos \lambda + dt v_x / V, r \sin \lambda + dt v_y / V),
\]

where \(V = \sqrt{v_x^2 + v_y^2} / V_c\) and \(V_c\) is the speed of the dog. After time \(dt\), the position of the dog is at \((x_c', y_c')\) as shown above, while the duck is at \((1, 2\pi \omega dt)\) (to the first order in \(dt\)). The distance \(\rho'\) between the dog and the duck after time \(dt\) is given as

\[
\rho' = \sqrt{(r \cos \lambda + dt v_x / V - 1)^2 + (r \sin \lambda + dt v_y / V - 2\pi \omega dt)^2}
\]

\[
= \sqrt{\rho^2 + \Delta \rho},
\]

where

\[
\Delta \rho = \{v_x^2 / V^2 + (v_y / V - 2\pi \omega)^2\} dt^2 + 2 \{v_x (r \cos \lambda - 1) / V + (v_y / V - 2\pi \omega) r \sin \lambda\} dt.
\]

Thus if we examine the sign of \(\Delta \rho\), we can judge whether the distance between duck and dog increases or decreases by the movement. When \(\Delta \rho > 0\), \(\rho\) increases, and when \(\Delta \rho < 0\), \(\rho\) decreases.

Figs. 3 to 6 show the spatial distribution of the sign of \(\Delta \rho\) as a function of the dog’s position \((x_c, y_c)\), with parameters set \(\omega = 1\), \(V_c = 1\), \(dt = 0.001\), and \(\tau_0 = 10, 100, 1000\). Here, the duck is at \((x_T, y_T) = (1, 0)\). The color at each point represents the absolute value of \(\Delta \rho\). They are shown separately for the cases \(\Delta \rho\) is positive or negative.

The duck moves into the positive \(y\) part of the plane starting from \((x_T, y_T) = (1, 0)\). Thus, it gives the tendency that \(\Delta \rho < 0\) if the dog is in the upper half of the plane indicating they get closer.

Without delay (Fig 3), the structure of distance change is quite orderly and smooth. However, as we increase delay with larger \(\tau_0\) (Fig. 4-6), it become rather complex structure showing that a small difference of the dog’s position relative to the duck can sensitively affect the increase or decrease of the distance.

Figure 3: \(|\Delta \rho|\) for the case of \(\tau_0 = 0\) plotted as a function of initial position of the chaser. Darker color reflects the smaller value of \(|\Delta \rho|\). (no data for white region.)
Figure 4: $|\Delta \rho|$ for the case of $\tau_0 = 10$ plotted as a function of initial position of the chaser. Darker color reflects the smaller value of $|\Delta \rho|$. (no data for white region.)

Figure 5: $|\Delta \rho|$ for the case of $\tau_0 = 100$ plotted as a function of initial position of the chaser. Darker color reflects the smaller value of $|\Delta \rho|$. (no data for white region.)

Figure 6: $|\Delta \rho|$ for the case of $\tau_0 = 1000$ plotted as a function of initial position of the chaser. Darker color reflects the smaller value of $|\Delta \rho|$. (no data for white region.)
Figure 7: $|\Delta \rho|$ for the case of $\tau_0 = 10000$ with non-moving target at $\left(1, 0\right)$, plotted as a function of initial position of the chaser. Darker color reflects the smaller value of $|\Delta \rho|$. (no data for white region.)

Also, as reference, we include the case where we assume the position of the duck did not move in $dt$ in Fig.7. Namely, the change of the distance between the moved dog in $dt$ and the point $(1, 0)$ for the case of $\tau_0 = 10000$. They look symmetric with respect to the $x$ axes as expected. This observation is another aspect of the complexity caused by the delay. With the larger delay, the distance between the chaser and the target does not change smoothly in agreement with the behavior observed in the previous section.

Conclusions

The detailed studies of the nature of the complex behaviors are left for the future. It is clear that even such a simple model with a distant–dependent delay can give rise to rich behavior. It will be of interest to see how delay can change the qualitative behavior of “group chase and escape” [1] as well.

References